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Semi-classical Quantization in N=4 Supersymmetric Yang-Mills Theory and Duality

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ABSTRACT

At a generic point in the moduli space of vacua of an $N = 4$ supersymmetric gauge theory with arbitrary gauge group the Higgs force does not cancel the magneto-static force between magnetic monopoles of distinct charge. As a consequence the moduli space of magnetically charged solutions is related in a simple way to those of the SU(2) theory. This leads to a rather simple test of S-duality. On certain subspaces of the moduli space of vacua the forces between distinct monopoles cancel and the test of S-duality becomes more complicated.

In this letter, we re-assess the evidence for S-duality in the spectrum of BPS states in $N = 4$ supersymmetric Yang-Mills theories with arbitrary gauge groups. Our conclusion is that at a generic point in the moduli space of vacua the spectrum of such states can be deduced from the spectrum of the SU(2) theory [1,2], although we cannot prove that it is complete. The Higgs field in both $N = 2$ and $N = 4$ supersymmetric gauge theories transforms as a vector of an internal global R-symmetry $\mathrm{SO}(N_R)$ and this complication will be seen to have profound implications for the spectrum of BPS states. The modifications to the semi-classical quantization of monopoles caused by having a vector Higgs field were considered in the context of an $N = 2$ theory in [3]. In this letter we consider these modifications in a theory with $N = 4$ supersymmetry. The essential modification in the vector Higgs model can be stated quite simply as the fact that between distinct static monopoles, the scalar interaction does not generically cancel the magneto-static interaction implying there are no static BPS solutions consisting of well separated distinct monopoles.

In order to compute the force between two well-separated monopoles, we shall follow the approach of [4,5] and effectively treat them as point particles. The Higgs field is denoted Φ^I, where the $\mathrm{SO}(N_R)$ vector index runs $I = 1, \ldots, 6$ in an $N = 4$ theory. The Higgs field of a single monopole behaves for large r as [3]

$$\Phi^I_{\boldsymbol{\alpha}} = \boldsymbol{a}^I \cdot \boldsymbol{H} - \frac{1}{er}\lambda^I_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}^\star \cdot \boldsymbol{H}), \qquad \lambda^I_{\boldsymbol{\alpha}} = \frac{\boldsymbol{a}^I \cdot \boldsymbol{\alpha}}{\parallel \boldsymbol{a}^I \cdot \boldsymbol{\alpha} \parallel}, \tag{1}$$

where $\Phi^I_0 = \boldsymbol{a}^I \cdot \boldsymbol{H}$ is the VEV of the Higgs field lying in some Cartan subalgebra of the Lie algebra g of the gauge group. This ensures that the Higgs VEV satisfies $[\Phi^I_0, \Phi^J_0] = 0$. The vector $\boldsymbol{\alpha}$ is some positive root of the Lie algebra and by definition the co-root is $\boldsymbol{\alpha}^\star = \boldsymbol{\alpha}/\boldsymbol{\alpha}^2$. In the above $\lambda^I_{\boldsymbol{\alpha}}$ is a unit N_R vector and $\parallel X^I \parallel = \sqrt{X^I X^I}$ denotes the length of the vector X^I. The solution has an asymptotic magnetic field

$$\vec{B} = \frac{\vec{r}}{er^3}\boldsymbol{\alpha}^\star \cdot \boldsymbol{H}, \tag{2}$$

and so $\boldsymbol{g} = \boldsymbol{\alpha}^\star$ is the vector magnetic charge of the monopole. The mass of the monopole is

$$M = \frac{4\pi}{e} \parallel \boldsymbol{a}^I \cdot \boldsymbol{\alpha} \parallel . \tag{3}$$

Consider the superposition of two such monopole solutions associated to positive roots $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$. For large separation the solution will be approximated by the linear superposition

$$\Phi^I = \Phi^I_0 + \Delta\Phi^I_{\boldsymbol{\gamma}} + \Delta\Phi^I_{\boldsymbol{\delta}}, \tag{4}$$

where $\Delta\Phi^I_{\boldsymbol{\alpha}} = \Phi^I_{\boldsymbol{\alpha}} - \Phi^I_0$. The potential due to long-range Higgs field can be deduced by considering the effect of the correction to the Higgs field by the second monopole on the first. The effect of the second monopole is to change the Higgs VEV at the first monopole to

$$\boldsymbol{a}^I \to \boldsymbol{a}^I - \frac{1}{er}\lambda^I_{\boldsymbol{\delta}}\boldsymbol{\delta}^\star. \tag{5}$$

Plugging this into the mass formula of the first monopole we deduce for static solutions that the leading order term in the potential describing the scalar interaction is

$$V_{\text{Higgs}} = -\frac{4\pi}{e^2 r}\boldsymbol{\gamma}^\star \cdot \boldsymbol{\delta}^\star \sum_{I=1}^{N_R} \lambda_{\boldsymbol{\gamma}}^I \lambda_{\boldsymbol{\delta}}^I. \tag{6}$$

The magneto-static scalar potential is coulomb-like and involves the inner product of the two vector magnetic charges:

$$V_{\text{em}} = \frac{4\pi}{e^2 r}\boldsymbol{\gamma}^\star \cdot \boldsymbol{\delta}^\star. \tag{7}$$

The total interaction is then

$$V_{\text{Higgs}} + V_{\text{em}} = \frac{4\pi}{e^2 r}\boldsymbol{\gamma}^\star \cdot \boldsymbol{\delta}^\star \left(1 - \sum_{I=1}^{N_R} \lambda_{\boldsymbol{\gamma}}^I \lambda_{\boldsymbol{\delta}}^I\right). \tag{8}$$

It follows that when $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are two distinct positive roots the net force is attractive if $\boldsymbol{\gamma} \cdot \boldsymbol{\delta} < 0$, and repulsive if $\boldsymbol{\gamma} \cdot \boldsymbol{\delta} > 0$. If $\boldsymbol{\gamma} \cdot \boldsymbol{\delta} \neq 0$ the force cancels only when $\lambda_{\boldsymbol{\gamma}}^I = \lambda_{\boldsymbol{\delta}}^I$. When $\boldsymbol{\gamma} \cdot \boldsymbol{\delta} = 0$ the two monopole do not interact at long range but there can be short range forces when the monopole cores overlap.

The consequences of this result are rather far-reaching. Spherically symmetric monopole solutions can be constructed whose magnetic charge is any co-root $\boldsymbol{\alpha}^\star$ of the Lie algebra. For theories with a single real Higgs field, Weinberg showed [6] that when $\boldsymbol{\alpha}$ is a non-simple root (with respect to a fundamental Weyl chamber defined by the Higgs VEV) then the solution could be always deformed away from spherical symmetry leading to an asymptotic configuration consisting of static constituent monopoles associated to the simple co-roots—the so-called 'fundamental' monopoles. In the single component Higgs model, the fact that these additional moduli appear is manifested in the fact that the overall long-range force between the constituent monopoles is always zero. On the contrary, in the vector Higgs model, the forces do not generically cancel. This is an indication that the additional moduli describing the degrees-of-freedom corresponding to separating the monopole into its fundamental constituents are not present and the original monopole is stable, and thus is itself 'fundamental'.

To confirm this heuristic picture, we will show that generically in the vector Higgs model, the moduli space of monopoles with magnetic charge $\boldsymbol{g} = n\boldsymbol{\alpha}^\star$, where $\boldsymbol{\alpha}$ is *any* root of g and n is a positive integer, is identical—up to a scale factor $1/\boldsymbol{\alpha}^2$—to the moduli space of n SU(2) monopoles. As a consequence, at most points in the moduli space of vacua GNO duality [7] is rather simple to test. In a nutshell, GNO duality states that a strongly coupled theory with gauge group G has an alternative description as a weakly coupled theory with gauge group $G^\star$—the group whose roots are the co-roots of G—such that the role of gauge bosons and monopoles is interchanged. The conjecture therefore leads to a prediction for the spectrum of monopoles which can be tested at weak coupling

using semi-classical techniques. Unlike [7], it is not necessary to consider the role of the centre of G since all the fields are adjoint-valued and space-time is Minkowskian.

The GNO duality conjecture is extended in the presence of a theta angle [1,8]. This 'S-duality' requires an infinite set of stable dyon states with quantum numbers $(\boldsymbol{g}, \boldsymbol{q}) = (n_m \boldsymbol{\alpha}^\star, n_e \boldsymbol{\alpha})$, where n_m and n_e are co-prime integers. The quantum number $\boldsymbol{q}$ determines the electric charge of the dyon state as we explain later. This prediction is a generalization of the SU(2) case [1]. As explained in [8], there is a simple way of determining whether the dyon $(n_m \boldsymbol{\alpha}^\star, n_e \boldsymbol{\alpha})$ corresponds under S-duality to a gauge boson with gauge group G or $G^\star$. Evidence for this extended duality conjecture now requires finding these additional dyon states—and no other ones—as stable bound states of monopoles. Stable dyon bound states correspond to harmonic forms on the centred monopole moduli space. For gauge group SU(2) the n-monopole moduli spaces have the form

$$\mathcal{M}_n = \mathbb{R}^3 \times \frac{S^1 \times \tilde{\mathcal{M}}_n^0}{\mathbb{Z}_n}. \tag{9}$$

The factors $\tilde{\mathcal{M}}_n^0$ are not explicitly known for $n \geq 3$, but evidence that they exhibit the relevant harmonic forms was found in [1,2].

In theories with larger gauge groups, we shall argue that the moduli space of a monopole with magnetic charge $\boldsymbol{\alpha}^\star$ is generically

$$\mathcal{M}_{\boldsymbol{\alpha}^\star} = \mathbb{R}^3 \times S^1, \tag{10}$$

i.e. the one monopole moduli space of the SU(2) theory. More generally, we have $\mathcal{M}_{n\boldsymbol{\alpha}^\star} \simeq \mathcal{M}_n$, up to a possible rescaling. However, on special subspaces in the space of vacua where certain monopoles reach the threshold for decay into their components, there is a discontinuous change:

$$\mathcal{M}_{\boldsymbol{\alpha}^\star} = \mathbb{R}^3 \times S^1 \to \mathbb{R}^3 \times \frac{\mathbb{R}^1 \times \mathcal{M}_{\text{rel}}}{\mathbb{Z}}, \tag{11}$$

to a higher dimensional space reflecting the fact that the monopole can dissociate into stable constituents. This change is only discontinuous to first order in the moduli space approximation. To higher orders, one should describe the change by introducing a potential on $\mathcal{M}_{\text{rel}}$.

On these special subspaces, GNO duality requires a single bound state associated to a unique normalisable harmonic form on $\mathcal{M}_{\text{rel}}$. This programme has been carried out with great success in the case of SU(n) gauge theory [5,9,10,11]. There has also been some analysis on subspaces when a non-abelian gauge symmetry is restored [12].

The mass formula for BPS states in $N = 4$ (or $N = 2$) supersymmetric Yang-Mills theory can be established by considering the energy:

$$U = \frac{1}{2} \int d^3x \, \mathrm{Tr} \left(\vec{E}^2 + \vec{B}^2 + \parallel D_0 \Phi^I \parallel^2 + \parallel \vec{D} \Phi^I \parallel^2 + \sum_{I<J} \left[\Phi^I, \Phi^J \right]^2 \right), \tag{12}$$

where we have assumed that the fermion fields are zero. This can be expressed in the following form:

$$U = \frac{1}{2}\int d^3x\,\mathrm{Tr}\left(\parallel \eta^I\vec{E}+\rho^I\vec{B}-\vec{D}\Phi^I \parallel^2 + \parallel D_0\Phi^I \parallel^2 + \sum_{I<J}\left[\Phi^I,\Phi^J\right]^2 \right.$$
$$\left. +2\sum_{I=1}^{N_R}\eta^I\vec{E}\cdot\vec{D}\Phi^I + 2\sum_{I=1}^{N_R}\rho^I\vec{B}\cdot\vec{D}\Phi^I\right), \tag{13}$$

where ρ^I and η^I are two orthonormal SO(N_R) vectors. From (13) we deduce a bound for the energy of a configuration:

$$U \geq \sum_{I=1}^{N_R}\left(\rho^I Q_M^I + \eta^I Q_E^I\right), \tag{14}$$

where Q_E^I and Q_M^I are defined by Gauss's law

$$Q_M^I = \int_{S^2_\infty} d\vec{S}\cdot\mathrm{Tr}\left(\vec{B}\Phi^I\right), \qquad Q_E^I = \int_{S^2_\infty} d\vec{S}\cdot\mathrm{Tr}\left(\vec{E}\Phi^I\right). \tag{15}$$

Here the integrals are taken over the sphere at spatial infinity.

The most stringent bound for the energy is achieved by maximizing the right-hand-side of (14) as a function of ρ^I and η subject to the fact that they are orthonormal. The solution is that ρ^I and η^I are in the plane defined by Q_M^I and Q_E^I as illustrated in Fig. 1. Here α is the angle between Q_M^I and Q_E^I and

$$\tan\theta = \frac{\parallel Q_E^I \parallel \cos\alpha}{\parallel Q_M^I \parallel + \parallel Q_E^I \parallel \sin\alpha}. \tag{16}$$

This gives the Bogomol'nyi Bound for the masses of particles of given electric and magnetic charges:

$$M^2 \geq \parallel Q_M^I \parallel^2 + \parallel Q_E^I \parallel^2 + 2\parallel Q_M^I \parallel\parallel Q_E^I \parallel \sin\alpha. \tag{17}$$

A configuration which saturates the bound, i.e. a BPS configuration, must satisfy the equations

$$D_0\Phi^I = 0, \qquad \eta^I\vec{E}+\rho^I\vec{B} = \vec{D}\Phi^I, \qquad [\Phi^I,\Phi^J] = 0. \tag{18}$$

The mass of a BPS state matches precisely what one expects from the $N=4$ super-algbera. This has two central charges $z_\pm$ expressed in terms of the electric and magnetic charges:

$$z_\pm^2 = \parallel Q_M^I \parallel^2 + \parallel Q_E^I \parallel^2 \pm 2\parallel Q_M^I \parallel\parallel Q_E^I \parallel \sin\alpha. \tag{19}$$

The mass of any state has to be greater than or equal to the greater of the two central charges, i.e. z_+ with our definitions. There are two types of BPS state. Firstly with

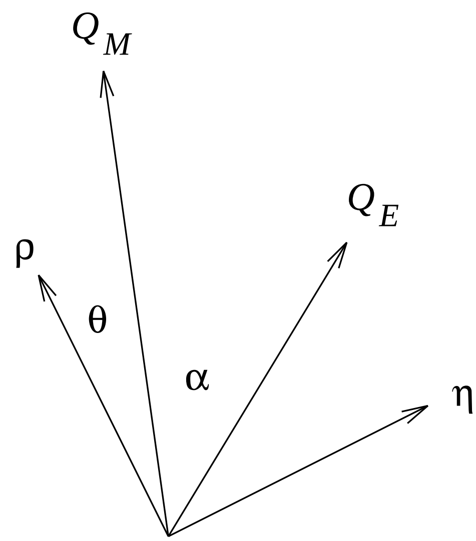

Figure 1. The relation between ρ^I, η^I, Q_M^I and Q_E^I

$M = z_+ > z_-$, which implies that $\alpha > 0$, i.e. $Q_M^I \not\propto Q_E^I$. Such states transform in 2^6 dimensional representations of $N = 4$ supersymmetry rather than the canonical 2^8 dimensional representation. If $\alpha = 0$, i.e. the electric and magnetic charge vectors are parallel $Q_M^I \propto Q_E^I$, then a BPS state with $M = z_\pm$ transforms in a 2^4 dimensional (ultra short) representation of $N = 4$ supersymmetry. In this case since $\alpha = 0$ these states have a mass

$$M^2 = \parallel Q_M^I \parallel^2 + \parallel Q_E^I \parallel^2 . \tag{20}$$

In this case the vectors ρ^I and η^I are only determined up to an SO(N_R) rotation around the common axis of Q_M^I and Q_E^I.

At spatial infinity the Higgs field is equal to its VEV $\Phi_0^I = \boldsymbol{a}^I \cdot \boldsymbol{H}$, and we can define the vector electric and magnetic charges via

$$Q_M^I = \frac{4\pi}{e} \boldsymbol{a}^I \cdot \boldsymbol{g}, \qquad Q_E^I = e \boldsymbol{a}^I \cdot \tilde{\boldsymbol{q}}. \tag{21}$$

With this normalization the vector magnetic charge $\boldsymbol{g}$ can be any vector of the co-root lattice of g. The generalized Dirac quantization condition, taking account the existence of a theta angle [13], leads to a quantization of vector electric charge:

$$\tilde{\boldsymbol{q}} = \boldsymbol{q} + \frac{\theta}{2\pi} \boldsymbol{g}, \tag{22}$$

where $\boldsymbol{q}$ is any vector of the root lattice of g. We choose to label BPS states with the vector quantum numbers $(\boldsymbol{g}, \boldsymbol{q})$.

The only known solutions of the Bogomol'nyi equations (18) are those that follow from embeddings of the SU(2) dyon solutions with a single real Higgs field and these solutions automatically have $Q_M^I \propto Q_E^I$, i.e. $\boldsymbol{q} \propto \boldsymbol{g}$. In [8], it was shown that a minimal set of states required by the conjectured form of S-duality have vector charges $(\boldsymbol{g}, \boldsymbol{q}) = (n_m \boldsymbol{\alpha}^\star, n_e \boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ is a root of g and n_m and n_e are two co-prime integers. These states certainly have $\boldsymbol{q} \propto \boldsymbol{g}$ and so $Q_E^I \propto Q_M^I$. They have a mass

$$M = e \left| n_e + \tau \frac{n_m}{\boldsymbol{\alpha}^2} \right| \parallel \boldsymbol{a}^I \cdot \boldsymbol{\alpha} \parallel, \tag{23}$$

and transform in ultra-short representations of $N = 4$ supersymmetry. In the above $\tau = \theta/2\pi + 4\pi i/e^2$. The middle-dimensional BPS multiplets with 2^6 states contain particles with spin > 1 and would presumably complicate the picture as far as duality is concerned and although there are no known solutions to the Bogomol'nyi equations with $Q_E^I \not\propto Q_M^I$ we cannot rule out their existence.

The corresponding situation in $N = 2$ supersymmetric gauge theories is rather different. In that case the supersymmetry algebra only has one central charge and there is only one type of BPS state which can have Q_M^I parallel to Q_E^I, or otherwise. In fact there *are* BPS states in these theories which do have $Q_E^I \not\propto Q_M^I$ arising from quantum corrections to the electric charge of a dyon [14,15]. This does not occur in the $N = 4$ theory because the quantum corrections vanish exactly.

One may wonder why the BPS states only have a vector magnetic charge $\boldsymbol{g} = n_m \boldsymbol{\alpha}^\star$, for some co-root $\boldsymbol{\alpha}^\star$, rather than *any* vector of the co-root lattice. At the present stage of understanding this is a conjecture which remains to be proved. In the semi-classical approximation one would first have to construct the moduli space of solutions to the Bogomol'nyi equations (18) with a given magnetic charge and then argue that only when $\boldsymbol{g} = n_m \boldsymbol{\alpha}^\star$ is there a bound-state corresponding to a stable BPS state. Some progress has been made in the construction of moduli spaces corresponding to solutions with magnetic charge which are not a multiple of a co-root in the SU(n) theory [10]. However these solutions exist in a theory with a single real Higgs field and they cannot be embedded in the vector Higgs model in any obvious way.

The moduli space of vacua $\mathcal{M}_{\rm vac}$ is parameterized by $\boldsymbol{a}^I$ modulo reflections by the Weyl group of G. We can use this freedom to fix $\sum_I \sigma^I \boldsymbol{\alpha}^I$ in the dominant Weyl chamber where σ^I is some arbitrary fixed vector. The vector $\sum_I \sigma^I \boldsymbol{\alpha}^I$ then defines a set of simple roots $\boldsymbol{\alpha}_i$, $i = 1, \ldots, \text{rank}(g)$, and fixes a notion of positive and negative roots. From the mass formula (23) it follows that BPS states are at the threshold for decay into other BPS states on certain submanifolds of $\mathcal{M}_{\rm vac}$. There are two possible types of decay. The first is familiar from the SU(2) theory where a state $(n_m \boldsymbol{\alpha}^\star, n_e \boldsymbol{\alpha})$ with n_m and n_e having a common factor p, $n_m = p n'_m$ and $n_e = p n'_e$, is at the threshold for decay into p states of charge $(n'_m \boldsymbol{\alpha}^\star, n'_e \boldsymbol{\alpha})$. Kinematically, these decays can occur at any point in $\mathcal{M}_{\rm vac}$. When the charges of a dyon are proportional to a non-simple root $\boldsymbol{\alpha}$ a second kind of decay can

occur. In that case there exist pairs of positive roots $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ such that

$$\boldsymbol{\alpha}^\star = N\boldsymbol{\gamma}^\star + M\boldsymbol{\delta}^\star, \tag{24}$$

for positive integers N and M, and the dyon can decay into a number of dyons of magnetic charge $\boldsymbol{\gamma}^\star$ and $\boldsymbol{\delta}^\star$ [3]. The important point is that kinematically this decay can only occur on a certain subspace of $\mathcal{M}_{\rm vac}$ of co-dimension 5 defined by the condition $\lambda^I_{\boldsymbol{\gamma}} = \lambda^I_{\boldsymbol{\delta}}$ which is precisely the condition that the force between the $\boldsymbol{\gamma}^\star$ and $\boldsymbol{\delta}^\star$ dyons vanishes. This defines the "Curve of Marginal Stability" (CMS) denoted $C_{\boldsymbol{\gamma},\boldsymbol{\delta}}$. There are cases in the non-simply-laced groups when $\boldsymbol{\gamma}\cdot\boldsymbol{\delta} = 0$ and there is no long-range force between the $\boldsymbol{\gamma}^\star$ and the $\boldsymbol{\delta}^\star$ dyons at any point in $\mathcal{M}_{\rm vac}$. Nevertheless, the $\boldsymbol{\alpha}^\star$ dyon is not generically at threshold for decay into the $\boldsymbol{\gamma}^\star$ and $\boldsymbol{\delta}^\star$ dyon. The resolution of this paradox is that there are generically no solutions of the Bogomol'nyi equations corresponding to separated $\boldsymbol{\gamma}^\star$ and $\boldsymbol{\delta}^\star$ dyons and the moduli space of BPS solutions of charge $\boldsymbol{\alpha}^\star$ is simply isomorphic to $\mathcal{M}_1$ describing the spherically symmetric $\boldsymbol{\alpha}^\star$ dyon.

In order to find explicit solutions to the Bogomol'nyi equations (18) consider the analogous equations for the theory with a single real Higgs field ϕ and gauge group SU(2):

$$D_0\phi = 0, \qquad \vec{E} = (\sin\xi)\,\vec{D}\phi, \qquad \vec{B} = (\cos\xi)\,\vec{D}\phi, \tag{25}$$

where $\tan\xi = Q_E/Q_M$ and Q_M and Q_E are defined as in (15) with Φ^I replaced by ϕ. These solutions have $Q_M = (4\pi v/e)n_m$, where n_m is an integer and v is the VEV of ϕ. The electric charge is subject to the Dirac quantization condition:

$$Q_E = ev\left(n_e + \frac{\theta}{2\pi}n_m\right), \qquad n_e \in \mathbb{Z}. \tag{26}$$

In order to write down solutions to (18) we pick a regular embedding of the Lie algebra $su(2)$ in the Lie algebra of the gauge group associated to a positive root $\boldsymbol{\alpha}$ and defined by the three generators $\{E_{\pm\boldsymbol{\alpha}}, \boldsymbol{\alpha}^\star\cdot\boldsymbol{H}\}$. We denote by $\phi^{\boldsymbol{\alpha}}$ and $A^{\boldsymbol{\alpha}}_\mu$ the solution of (25) embedded in the theory with a larger gauge group using this $su(2)$ Lie subalgebra. The ansatz for the solution of the full Bogomol'nyi equations (18) is

$$\Phi^I = \lambda^I_{\boldsymbol{\alpha}}\phi^{\boldsymbol{\alpha}} + \big(\boldsymbol{a}^I - (\boldsymbol{a}^I\cdot\boldsymbol{\alpha}^\star)\boldsymbol{\alpha}\big)\cdot\boldsymbol{H}, \qquad A_\mu = A^{\boldsymbol{\alpha}}_\mu, \tag{27}$$

where the VEV of the SU(2) Higgs field is $v = \parallel \boldsymbol{a}^I\cdot\boldsymbol{\alpha} \parallel$ and $\lambda^I_{\boldsymbol{\alpha}}$ is defined in (1). This ensures that the embedded solution (27) has the correct VEV. The solution has $(\boldsymbol{g},\boldsymbol{q}) = (n_m\boldsymbol{\alpha}^\star, n_e\boldsymbol{\alpha})$ and in particular $Q^I_M \propto Q^I_E$.

Using this construction we can find purely magnetically charged solutions with vector magnetic charge $\boldsymbol{g} = n_m\boldsymbol{\alpha}^\star$. The moduli space of these solutions can be probed locally by finding the zero-modes for fluctuations in the Higgs and gauge field. Equivalently, it

is more convenient to find the Dirac zero-modes since they are paired with the bosonic modes by unbroken supersymmetries [16].

In the appendix, we show that the zero-mode equation can be cast in a form familiar from the real Higgs theory [6]. It is convenient to define two orthogonal projections of Φ^I with respect to the vector $\lambda^I_{\boldsymbol{\alpha}}$:

$$\Phi = \sum_{I=1}^{N_R} \lambda^I_{\boldsymbol{\alpha}} \Phi^I \quad \text{and} \quad \hat{\Phi}^I = \Phi^I - \lambda^I_{\boldsymbol{\alpha}} \sum_{J=1}^{N_R} \lambda^J_{\boldsymbol{\alpha}} \Phi^J. \tag{28}$$

Φ is then Weinberg's ansatz for the monopole solutions in the real Higgs model [6] with VEV $\sum_I \lambda^I_{\boldsymbol{\alpha}} \boldsymbol{a}^I$. The orthogonal projection is a constant:

$$\hat{\Phi}^I = \hat{\boldsymbol{a}}^I \cdot \boldsymbol{H} = \left(\boldsymbol{a}^I - \lambda^I_{\boldsymbol{\alpha}} \sum_{J=1}^{N_R} \lambda^J_{\boldsymbol{\alpha}} \boldsymbol{a}^J \right) \cdot \boldsymbol{H}. \tag{29}$$

We are now in a position to describe the zero modes. The idea is to consider the relation between the corresponding zero modes for the real Higgs theory [6]. One first expands the adjoint-valued modes in a Cartan-Weyl basis. There are four zero-modes taking values in the $su(2)$ Lie algebra $\{E_{\pm\boldsymbol{\alpha}}, \boldsymbol{\alpha}^\star \cdot \boldsymbol{H}\}$ corresponding to overall translations and charge rotations. The remaining generators $E_{\boldsymbol{\beta}}$ can be grouped into representations under the embedding SU(2) labelled by the total isospin t. The member $E_{\boldsymbol{\beta}}$ with the lowest value of $t_3 = \boldsymbol{\alpha} \cdot \boldsymbol{\beta}/\boldsymbol{\alpha}^2 = -t$ can be used to label each isospin multiplet. The zero modes associated to each pair of isospin multiplets containing $E_{\pm\boldsymbol{\beta}}$ describe the freedom for the dyon to decay as in (24) with $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ being expressed in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ [3]. This means that the definition of the CMS $C_{\boldsymbol{\gamma},\boldsymbol{\delta}}$ is equivalent to $\hat{\boldsymbol{a}}^I \cdot \boldsymbol{\beta} = 0$, $\forall I$. The computation in the appendix shows that these zero-modes of the real Higgs theory are lifted in the vector Higgs theory unless they are annihilated by $\hat{\Phi}^I$. It follows immediately that at a generic point in the moduli space which is not on a CMS all zero-modes, except those corresponding to translations and charge rotations, are lifted. Hence generically the spherically symmetric monopole with $\boldsymbol{g} = \boldsymbol{\alpha}^\star$ will be stable. In the case of higher magnetic charges the zero-modes associated to the root $\boldsymbol{\alpha}$ can be identified with the zero-modes of the SU(2) n_m-monopole solution since they all take values in the embedding $su(2)$. This proves that the monopole moduli space of charge $\boldsymbol{g} = n_m \boldsymbol{\alpha}^\star$ is identical to the SU(2) charge n_m monopole moduli space up to a scale factor $1/\boldsymbol{\alpha}^2$ from the normalization of the killing form. After quantization the spectrum of stable BPS dyons have charges $(\boldsymbol{g}, \boldsymbol{q}) = (n_m \boldsymbol{\alpha}^\star, n_e \boldsymbol{\alpha})$ where (n_m, n_e) are co-prime integers.

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Appendix

In this appendix we study the Dirac equation in the background of the classical fields (27) and show how to relate the zero-modes of the single Higgs to the vector Higgs model. A convenient way to obtain the $d = 4$ $N = 4$ Dirac equation is by dimensional reduction from the $d = 10$ $N = 1$ Dirac equation. We follow closely the conventions of Osborn [17], with minor notational changes. The $d = 4$ $N = 4$ supermultiplet of adjoint valued fields contains, as well as the gauge field and the six scalars Φ^I, four Majorana fermions ψ^a_α, where $a = 1, \ldots, 4$ denotes the $\mathrm{SO}(6)_R$ (or more correctly SU(4)) spinor index, and α denotes the usual Dirac spinor index.

The Dirac equation obtained by dimensionally reducing $i\Gamma^A D_A \psi = 0$ is

$$\left\{ i\gamma^\mu D_\mu - i\alpha^m \Phi^m + \gamma_5 \beta^{\dot{n}} \Phi^{3+\dot{n}} \right\} \psi = 0, \tag{A.1}$$

where $m, \dot{n} = 1, 2, 3$ and the gauge and Higgs fields act by adjoint action on ψ. The γ-matrices carry space-time spinor indices, while the 4×4 matrices α^m and $\beta^{\dot{n}}$ carry $\mathrm{SO}(6)_R$ spinor indices. They are antisymmetric, (anti-)self-dual matrices with real entries satisfying

$$\begin{gathered} \{\alpha^m, \alpha^n\} = -2\delta^{mn} \qquad \{\beta^{\dot{m}}, \beta^{\dot{n}}\} = -2\delta^{\dot{m}\dot{n}} \\ \left[\alpha^m, \alpha^n\right] = -2\epsilon^{mnp}\alpha^p \qquad \left[\beta^{\dot{m}}, \beta^{\dot{n}}\right] = -2\epsilon^{\dot{m}\dot{n}\dot{p}}\beta^{\dot{p}} \\ \left[\alpha^m, \beta^{\dot{n}}\right] = 0. \end{gathered} \tag{A.2}$$

In a static background we look for stationary solutions $\psi(\vec{r}, t) = e^{-iEt}\psi(\vec{r})$, leading to the the Dirac Hamiltonian equation

$$\left\{ i\gamma^0 \gamma_i D_i - i\gamma^0 \alpha^m \Phi^m - \gamma^0 \gamma_5 \beta^{\dot{n}} \Phi^{3+\dot{n}} \right\} \psi = E\psi. \tag{A.3}$$

Squaring this we find

$$\left\{ -D_i^2 + \parallel \Phi^I \parallel^2 + \frac{1}{2}\gamma_{ij} F_{ij} - \left(\gamma_i \lambda^m \alpha^m + i\gamma_i \gamma_5 \lambda^{3+\dot{n}} \beta^{\dot{n}} \right) B_i \right\} \psi = E^2 \psi, \tag{A.4}$$

where $\gamma_{ij} = i\epsilon_{ijk}\gamma_0\gamma_5\gamma_k$. Now introduce the following set of Euclidean γ-matrices:

$$\begin{aligned} \tilde{\gamma}_i &= \gamma_0 \gamma_i, \qquad \tilde{\gamma}_4 = \gamma_0 \lambda^m \alpha^m + i\gamma_0 \gamma_5 \lambda^{3+\dot{n}} \beta^{\dot{n}} \\ \tilde{\gamma}_5 &= \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \tilde{\gamma}_4 = -i\gamma_0 \gamma_5 \lambda^m \alpha^m + \gamma_0 \lambda^{3+\dot{n}} \beta^{\dot{n}}. \end{aligned} \tag{A.5}$$

This gives

$$\left\{ -D_i^2 + \parallel \Phi^I \parallel^2 - i\gamma_5 \tilde{\gamma}_i B_i \left(1 + \tilde{\gamma}_5\right) \right\} \psi = E^2 \psi \tag{A.6}$$

Define the chiral projections $\psi_\pm = \frac{1}{2}\{1 \pm \tilde{\gamma}_5\}\psi$ and a representation of the Pauli matrices $\sigma_i = \gamma_5\tilde{\gamma}_i$. Using this (A.6) becomes

$$\left(-D_i^2 + \parallel \Phi^I \parallel^2 - 2i\vec{\sigma}\cdot\vec{B}\right)\psi_+ = E^2\psi_+, \qquad \left(-D_i^2 + \parallel \Phi^I \parallel^2\right)\psi_- = E^2\psi_-. \tag{A.7}$$

Introducing the projections of Φ^I defined in (28): $\parallel \Phi^I \parallel^2 = \Phi^2 + \parallel \hat{\Phi}^I \parallel^2$, and defining

$$\mathcal{D} = -i\vec{\sigma}\cdot\vec{D} - i\Phi, \qquad \mathcal{D}^* = -i\vec{\sigma}\cdot\vec{D} + i\Phi, \tag{A.8}$$

we have

$$\mathcal{D}^*\mathcal{D} = -D_i^2 + \Phi^2 - 2i\vec{\sigma}\cdot\vec{B}, \qquad \mathcal{D}\mathcal{D}^* = -D_i^2 + \Phi^2. \tag{A.9}$$

In terms of these operators (A.7) becomes

$$\left(\mathcal{D}^*\mathcal{D} + \parallel \hat{\Phi}^I \parallel^2\right)\psi_+ = E^2\psi_+, \qquad \left(\mathcal{D}\mathcal{D}^* + \parallel \hat{\Phi}^I \parallel^2\right)\psi_- = E^2\psi_-. \tag{A.10}$$

$\mathcal{D}\mathcal{D}^*$ is a positive definite operator and has no non-trivial zero-modes. The number of zero-modes of $\mathcal{D}^*\mathcal{D}$ is determined in [6]. These will only survive as zero-modes of (A.3) if annihilated by $\hat{\Phi}^I$.

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